Week 13 Adjoint operator

Today: Theory of operators

Riesz's theorem ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$)

Let H be a Hilbert space

$f: H \to \mathbb{F}$ is a bounded linear functional

Then $\exists$ unique $z \in H$ such that

$$f(x) = \langle x, z \rangle \quad \forall x \in H$$

Also, $\|f\| = \|z\|$

$\therefore$ Bounded linear functionals on Hilbert space are easy to describle

Pf ① Existence of z

Case 1: $f = 0$, take $z = \vec{0}$

Then $\forall x \in H$, $f(x) = 0 = \langle x, \vec{0} \rangle$

Case 2: If $f \neq 0$

Consider $N(f) = \{ x \in H : f(x) = 0 \}$

$f \neq 0 \Rightarrow N(f) \neq H$

f is bounded $\Leftrightarrow$ continuous

$\Rightarrow N(f)$ is closed subspace of H

$$\Rightarrow \quad H = \underset{\neq H}{\underset{\uparrow}{N(f)}} \oplus \underset{\Rightarrow N(f)^\perp \neq \{0\}}{\underset{\uparrow}{N(f)^\perp}}$$

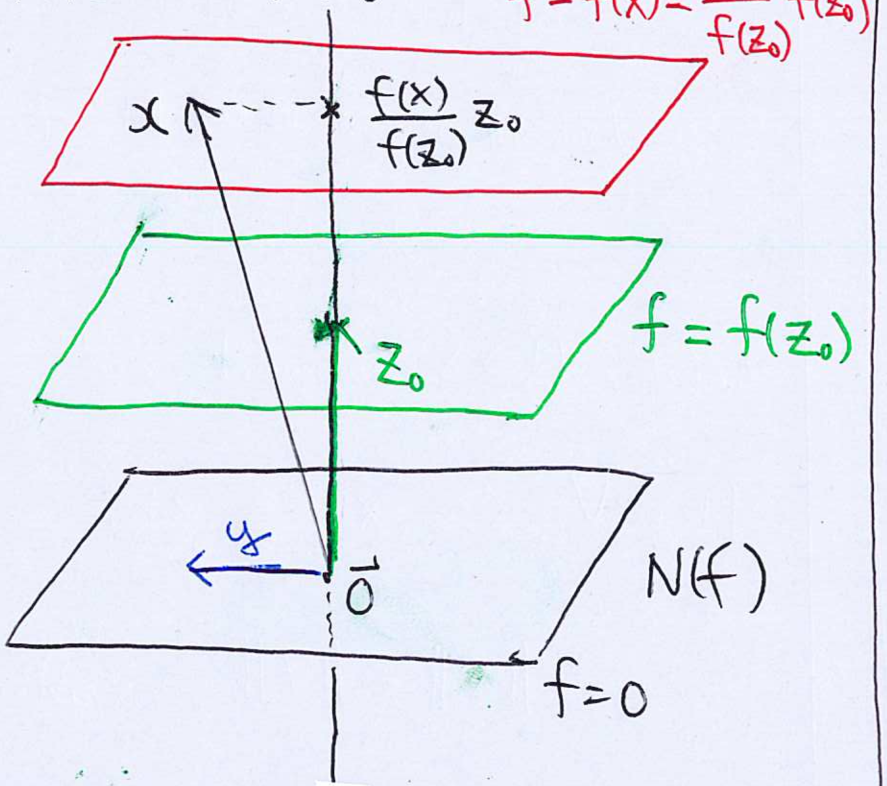
Take $z_0 \in N(f)^{\perp}$

We can assume $\|z_0\| = 1$

(if not, do normalization)

$$f\left(\frac{f(x)}{f(z_0)} z_0\right) = \frac{f(x)}{f(z_0)} f(z_0)$$

Note $f(z_0) \neq 0$



Define $z = \overline{f(z_0)}\, z_0$

For any $x \in H$

Let $y = x - \frac{f(x)}{f(z_0)} z_0$

Then
$$
\begin{aligned}
f(y) &= f(x) - f\left(\frac{f(x)}{f(z_0)} z_0\right) \\
&= f(x) - \frac{f(x)}{f(z_0)} f(z_0) \\
&= f(x) - f(x) \\
&= 0
\end{aligned}
$$

$\Rightarrow y \in N(f)$

$\Rightarrow \langle y, z_0 \rangle = 0$

$\Rightarrow \left\langle x - \frac{f(x)}{f(z_0)} z_0 , z_0 \right\rangle = 0$

$$\Rightarrow \langle x, z_0\rangle - \frac{f(x)}{f(z_0)}\langle z_0, z_0\rangle = 0$$

$$\Rightarrow \langle x, z_0\rangle = \frac{f(x)}{f(z_0)}$$

$$\Rightarrow f(x) = f(z_0)\langle x, z_0\rangle$$

$$= \langle x, \overline{f(z_0)}\, z_0\rangle$$

$$= \langle x, z\rangle$$

② Uniqueness of z

Suppose $\exists\, z_1, z_2 \in H$ such that

$$\langle x, z_1\rangle = f(x) = \langle x, z_2\rangle \quad \forall x \in H$$

$$\Rightarrow \langle x, z_1\rangle - \langle x, z_2\rangle = 0$$

$$\Rightarrow \langle x, z_1 - z_2\rangle = 0$$

Take $x = z_1 - z_2$

$$\Rightarrow \langle z_1 - z_2, z_1 - z_2\rangle = 0$$

$$\Rightarrow z_1 - z_2 = 0 \Rightarrow z_1 = z_2$$

$\Rightarrow$ Uniqueness

If $f=0$, then $z=0$, $\|f\| = \|z\| = 0$

if $f \neq 0$, $z \neq 0$,

③ $$\|z\|^2 = \langle z, z\rangle = f(z) \le \|f\|\,\|z\|$$

$$\Rightarrow \|z\| \le \|f\|$$

Next, take $x \in H$, $\|x\| = 1$, then

$$|f(x)| = |\langle x, z\rangle| \le \|x\|\,\|z\| \quad \text{(Cauchy-Schwarz)}$$

$$= \|z\|$$

$$\Rightarrow \|f\| \le \|z\|$$

$$\Rightarrow \|f\| = \|z\|$$

Next do it for an operator:

Defn Let H_1, H_2 be Hilbert spaces

$T: H_1 \to H_2$ be a linear operator.

A linear operator $T^*: H_2 \to H_1$ is said to the Hilbert-adjoint (adjoint) of T if

$$\underbrace{\langle T(x), y \rangle}_{\text{inner product on } H_2} = \underbrace{\langle x, T^*(y) \rangle}_{\text{inner product on } H_1} \quad \forall x \in H_1, \ y \in H_2$$

Rmk ① Not all operators have adjoint

② By taking conjugate,

$$\langle T^*(y), x \rangle = \langle y, T(x) \rangle$$

Thm 3.9-2 If $T: H_1 \to H_2$ is bounded then T^* exists. and unique.

Also, T^* is bounded and $\|T^*\| = \|T\|$

Pf ① Define T^*:

Let $y \in H_2$, consider $f_y: H_1 \to \mathbb{F}$ defined by $f_y(x) = \langle T(x), y \rangle$

Then ⓐ f_y is linear $\left(\begin{array}{l} T \text{ is linear} \\ \text{1st component of inner} \\ \text{produt is linear} \end{array}\right)$

ⓑ $|f_y(x)| = |\langle T(x), y \rangle|$

$$\leq \|T(x)\| \, \|y\|$$

$$\leq (\|T\| \, \|y\|) \, \|x\|$$

$\Rightarrow f_y$ is bounded

Riesz theorem $\Rightarrow \exists Z_y \in H_1$ such that

$$f_y(x) = \langle x, Z_y \rangle \quad \forall x \in H_1$$

Define $T^*(y) = Z_y$, then

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle \quad \forall x \in H_1$$

2 Show that T^* is unique

Suppose $S: H_2 \to H_1$ such that

$$\langle x, S(y) \rangle = \langle T(x), y \rangle = \langle x, T^*(y) \rangle$$

$$\Rightarrow \langle x, S(y) - T^*(y) \rangle = 0 \quad \forall x, y$$

$$\Rightarrow S(y) - T^*(y) = 0 \quad \forall y \quad \Rightarrow S = T^*$$

$\Rightarrow$ uniqueness

3 Show that T^* is linear

Let $y_1, y_2 \in H_2$, $\alpha \in \mathbb{F}$. Then $\forall x \in H_1$

$$\begin{aligned}\langle x, T^*(y_1 + \alpha y_2) \rangle &= \langle T(x), y_1 + \alpha y_2 \rangle \\ &= \langle T(x), y_1 \rangle + \bar{\alpha} \langle T(x), y_2 \rangle \\ &= \langle x, T^*(y_1) \rangle + \bar{\alpha} \langle x, T^*(y_2) \rangle \\ &= \langle x, T^*(y_1) + \alpha T^*(y_2) \rangle\end{aligned}$$

x is arbitrary

$$\Rightarrow T^*(y_1 + \alpha y_2) = T^*(y_1) + \alpha T^*(y_2)$$

$\Rightarrow T^*$ is linear

④ Show that T^* is bounded, $\|T\| = \|T^*\|$

$\forall y \in H_2$, $\|y\| = 1$

$$\|T^*(y)\|^2 = \langle T^*(y), T^*(y) \rangle$$
$$= \langle TT^*(y), y \rangle$$
$$\leq \|TT^*(y)\| \, \|y\|$$
$$\leq \|T\| \, \|T^*(y)\|$$

$\Rightarrow \|T^*(y)\| \leq \|T\| \quad (\because \|T^*(y)\| \geq 0)$

y is arbitrary with $\|y\| = 1$

$\Rightarrow \|T^*\| \leq \|T\|$ and T^* is bonded

Similarly, $\|T\| \leq \|T^*\| \Rightarrow \|T\| = \|T^*\|$

<u>Examples of adjoint Operator</u>

① Shifting operators:

Define $R, L : \ell^2 \to \ell^2$

$$R(x_1, x_2, x_3 \cdots) = (0, x_1, x_2, x_3, \cdots)$$
$$L(y_1, y_2, y_3, \cdots) = (y_2, y_3, y_4, \cdots)$$

$$\langle R(\vec{x}), \vec{y} \rangle = \sum_{i=1} x_i \overline{y_{i+1}}$$
$$\langle x, L(\vec{y}) \rangle = \sum_{i=1} x_i \overline{y_{i+1}}$$

$\Rightarrow \langle R(\vec{x}), \vec{y} \rangle = \langle x, L(\vec{y}) \rangle$

$\Rightarrow R^* = L$

2. Let $A \in M_{m \times n}(\mathbb{C})$ — m rows, n columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Define $A^* = \overline{A}^t$ (conjugate transpose)

$$A^* = \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{m1}} \\ \overline{a_{12}} & \overline{a_{22}} & & \overline{a_{m2}} \\ \vdots & \vdots & & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & & \overline{a_{mn}} \end{bmatrix}$$

Let $T : \mathbb{C}^n \to \mathbb{C}^m$

$$T(x) = Ax \in \mathbb{C}^m$$

Here $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^n$

Then $\forall x \in \mathbb{C}^n,\ y \in \mathbb{C}^m$

$$\begin{aligned} \langle T(x), y \rangle &= \langle Ax, y \rangle \\ &= \sum_i \Big(\sum_j a_{ij} x_j \Big) \overline{y_i} \\ &= \sum_j x_j \Big(\sum_i a_{ij} \overline{y_i} \Big) \\ &= \sum_j x_j \overline{\Big(\sum_i \overline{a_{ij}}\, y_i \Big)} \\ &= \langle x, A^* y \rangle \end{aligned}$$

$\Rightarrow T^*(y) = A^* y$

Adjoint $\longleftrightarrow$ Conjugate transpose

3 Let $X = \left\{ f : \mathbb{R} \to \mathbb{R} : f \text{ is smooth } 2\pi\text{-periodic} \right\}$

Define $\langle f, g \rangle = \int_0^{2\pi} f(x) g(x)\, dx$

$T : X \to X$ by $Tf = f'$

Then $\langle Tf, g \rangle$

$$= \int_0^{2\pi} f'(x) g(x)\, dx$$

$$= \left[f(x) g(x) \right]_0^{2\pi} - \int_0^{2\pi} f(x) g'(x)\, dx$$

$$= f(2\pi) g(2\pi) - f(0) g(0) - \langle f, Tg \rangle$$

$$= -\langle f, Tg \rangle = \langle f, -Tg \rangle$$

$\Rightarrow T^* = -T$

Rmk

X is not complete

Note $(T^2)^* = T^2$ is self-adjoint

Prop of adjoint

Let H_1, H_2 be Hilbert space, α is scalar

$S, T : H_1 \to H_2$ be bounded. Then

1. $(S+T)^* = S^* + T^*$
2. $(\alpha T)^* = \bar{\alpha} T^*$
3. $(T^*)^* = T$
4. $\|T^*T\| = \|TT^*\| = \|T\|^2$ (Compare $\|T^2\| \le \|T\|^2$)
5. $T^*T = 0 \Leftrightarrow T = 0$
6. If $H_1 = H_2$, then $(ST)^* = T^* S^*$

Pf of 1

Let $x \in H_1$, $y \in H_2$

$$\langle x, (S+T)^*(y)\rangle = \langle (S+T)(x), y\rangle$$
$$= \langle S(x)+T(x), y\rangle$$
$$= \langle S(x), y\rangle + \langle T(x), y\rangle$$
$$= \langle x, S^*(y)\rangle + \langle x, T^*(y)\rangle$$
$$= \langle x, S^*(y)+T^*(y)\rangle$$

x is arbitrary

$\Rightarrow (S+T)^*(y) = S^*(y)+T^*(y) \ \forall y$

$\Rightarrow$ 1

Pf of 4

Let $x \in H_1$, $\|x\|=1$, then

$$\|T(x)\|^2 = \langle T(x), T(x)\rangle$$
$$= \langle x, T^*T(x)\rangle$$
$$\leq \|x\|\,\|T^*T\|\,\|x\| = \|T^*T\|$$

x is arbitrary with $\|x\|=1$

$\Rightarrow \|T\|^2 \leq \|T^*T\|$

Other direction: Exercise

(Easy: $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|\|T\| = \|T\|^2$